ON THE TENSOR METHOD IN THE THEORY OF NONSTEADY SPATIAL FLOWS OF THE "DOUBLE WAVE" TYPE

(O TENZORNOM METODE V TEORII NEUSTANOVIVSHIKHSIA PROSTRANSTVENNYKH TECHENII TIPA "DVOINOI VOLNY")

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This investigation is concerned with problems of the broad class of nonsteady spatial irrotational flows of a compressible gas, which may be solved successfully by using the technique of tensor calculation in four-dimensional space. Under investigation are only certain general "intrinsic" properties of double waves, i.e. those properties which do not depend on special conditions of the motion, such as initial and boundary conditions.

1. We shall assume that in the region of flow under consideration there are no strong discontinuities (shock-waves) and, also, neglecting the viscosity and the heat conductivity of the gas, we shall assume the motion to be isentropic; finally, in the equations of motion we shall omit outside forces (as is done, incidentally, usually in all typical problems of gasdynamics). Under these conditions it is the customary practice to limit the investigations to irrotational (potential) flows.

We shall introduce the four-dimensional Euclidian space – the time R_4 being the fourth coordinate – which henceforth we shall call the space of the motion. In this space an orthogonal system of Cartesian coordinates will serve as a system of reference:

$$x^1 \equiv x, \qquad x^2 \equiv y, \qquad x^3 \equiv z, \qquad x^4 \equiv v^\circ t \tag{a}$$

where v° is an arbitrary constant with the dimension of a velocity and t is the time.

Thereupon, starting with the three-dimensional velocity vector of the gas particles $\mathbf{v}$ with components along the axes x^1, x^2, x^3, which are equal to v_1, v_2, v_3, respectively, we shall introduce in this space of four-dimensional motion the four-dimensional field vector $\mathbf{u}$, defined by the equations

$$u_k = \begin{cases} v_k, & \text{if } k = 1, 2, 3 \\ v^\circ = \text{const}, & \text{if } k = 4 \end{cases} \tag{1.1}$$

Using this field vector **u** the equation of continuity may be represented in abbreviated tensor notation* as follows:

$$\partial (\rho u^k)/\partial x^k = 0 \tag{1.2}$$

where $\rho(x^1, x^2, x^3, x^4)$ is the gas density. In this equation and in the other formulas, for which the orthogonal Cartesian coordinate system serves as a system of reference, the difference between the contra- and covariant vector components is purely formal: $u^k = u_k$. The necessity to differentiate them sharply arises from the moment we introduce general curvilinear coordinates (see Section 3).

Emphasis is laid here upon the symmetrical form of Equation (1.2) with respect to all coordinates x^k. In contradistinction, the structure of the three equations of motion for an isentropic compressible ideal gas, even after using the condition of irrotationality of the flow

$$\partial v_i/\partial x^j = \partial v_j/\partial x^i \qquad (i, j = 1, 2, 3) \tag{1.3}$$

is absolutely different with respect to time and the spatial coordinates

$$v^\circ (\partial v_i/\partial x^4) + \partial H/\partial x^i = 0 \qquad (i = 1, 2, 3) \tag{1.4}$$

where H is the "total enthalpy"

$$H \equiv \tfrac{1}{2} (v)^2 + \int_0^\rho \frac{c^2}{\rho} d\rho = \tfrac{1}{2} (v)^2 + \frac{c^2}{\varkappa - 1} \tag{1.5}$$

where c is the local sound velocity and κ is the isentropic index,

$$(v)^2 = |\mathbf{v}|^2 = v_i v^i$$

Note that for steady flows the asymmetry of Equations (1.4) vanishes, because in this case $\partial v_i/\partial x^4 \equiv 0$. At the same time we obtain immediately the first integral of these equations in the form of Bernoulli's

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* According to this notation, every expression in which any letter index appears twice - one time as a contravariant (superior) and the second time as a covariant index (inferior) - is to be summed over all the values of this index. Let it be agreed that $i, j = 1, 2, 3$; whereas $k, l, m = 1, 2, 3, 4$.

equation, $H =$ const. Under these conditions we may eliminate the density ρ from the equation of continuity (1.2) without much difficulty, arriving thereby at the known differential equation of steady flow

$$(c^2 \delta_j^i - v^i v_j)\frac{\partial v^j}{\partial x^i} = 0, \qquad \delta_j^i = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases} \tag{1.6}$$

It is necessary only to substitute in (1.6) the covariant derivatives for the partial derivatives, so that the equation will assume a covariant form.

2. In the case of nonsteady flow the density, pressure, sound velocity and other gasdynamic quantities will not be functions of the velocity only. Therefore, the study of flows which are not steady, requires a quite substantial generalization of the vector space of the velocities $\boldsymbol{v}$ through the addition of a fourth component which would also depend on the density ρ (or on the pressure p or sound velocity c). Through that generalization, the equations of motion (1.4) may be made symmetrical. Indeed, every one of these three equations contains a corresponding derivative of the "total enthalpy" H. Therefore, it is clear that when introducing a new generalized (four-dimensional) Euclidian vector space it is sufficient to assume (in right-angle Cartesian coordinates)

$$w_k = \begin{cases} v_k & (k = 1, 2, 3) \\ -H/v^\circ & (k = 4) \end{cases} \tag{2.1}$$

in order to obtain, instead of Equations (1.3) and (1.4), the equivalent and already wholly symmetrical system of relationships

$$\frac{\partial w_m}{\partial x^k} = \frac{\partial w_k}{\partial x^m} \qquad (k, m = 1, 2, 3, 4) \tag{2.2}$$

From these equations it follows that the four-dimensional vector field ω introduced here has a potential, whereas usually the velocity potential $\phi(x^1, x^2, x^3, x^4)$ serves as a potential function of the flow

$$w_k = \partial\varphi/\partial x^k \qquad (k = 1, 2, 3, 4) \tag{2.3}$$

Now we may eliminate the density ρ from the equation of continuity. Differentiating Equation (1.5) with respect to x^l and taking into account (1.1) and (2.1), we find

$$u^m \frac{\partial w_m}{\partial x^l} + \frac{c^2}{\rho}\frac{\partial \rho}{\partial x^l} = 0 \qquad (l = 1, 2, 3, 4) \tag{2.4}$$

Then, eliminating from (1.2) and (2.4) the density ρ and its derivatives, we arrive at the equation

$$c^2 \frac{\partial u^k}{\partial x^k} - u^l u^m \frac{\partial w_m}{\partial x^l} = 0 \tag{2.5}$$

If we write this equation in an expanded form and introduce the velocity potential $\phi(x^1, x^2, x^3, x^4)$, then we shall immediately obtain the well-known nonlinear differential equation in terms of partial derivatives of the second order, by which are governed the very general potential flows of a compressible gas.

In order to obtain in due course a very important transformation of this equation, we shall note, first of all, that according to (1.1) and (2.1) we have

$$\frac{\partial u^k}{\partial x^k} = \tau^{lm} \frac{\partial w_m}{\partial x^l} \tag{2.6}$$

where the tensor of the second rank τ^{lm} in the chosen orthogonal Cartesian coordinate system (x^k) is determined by the matrix

$$||\tau^{lm}|| = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix} \tag{2.7}$$

Introducing also the symmetric tensor

$$T^{lm} \equiv c^2 \tau^{lm} - u^l u^m \qquad (l, m = 1, 2, 3, 4) \tag{2.8}$$

Equation (2.5) may be written in the particularly simple and short tensor form

$$T^{lm} \frac{\partial w_m}{\partial x^l} = 0 \tag{2.9}$$

Note that in the coordinate system chosen (x^k) the component $\tau^{44} = 0$. In this way the tensor τ^{lm}, in distinction to δ^i_j of (1.6), is not a simple unit tensor of the four-dimensional space of motion R, since its structure is more complicated.

From this situation, particularly, stem the difficulties discussed at the very beginning (Section 1), and here lies also the key to their solution.

3. We shall turn now from the rectilinear system of reference to an arbitrary system of curvilinear coordinates (x^λ); we shall denote the curvilinear coordinates by Greek indices.

To accomplish this, we shall substitute everywhere, in strict compliance with the known rule on the location of contra- and covariant indices in Equation (2.9), the partial derivatives with respect to the Cartesian coordinates $\partial w_m / \partial x^l$ by the covariant derivatives $\nabla_\lambda w_\mu$. Thus, we obtain

$$T^{\lambda\mu} \nabla_\lambda w_\mu = 0 \qquad (T^{\lambda\mu} \equiv c^2 \tau^{\lambda\mu} - u^\lambda u^\mu) \tag{3.1}$$

Here

$$\nabla_\lambda w_\mu = \frac{\partial w_\mu}{\partial x^\lambda} - \Gamma^\nu{}_{\lambda\mu} w_\nu \tag{3.2}$$

where $\Gamma_{\lambda\mu}{}^\nu$ denotes the Christoffel symbol of the second order.

4. Every solution w of Equation (2.9) or (3.1) may be interpreted as a mapping of the space of motion as a whole, or of any particular part of it, into some region of the four-dimensional Euclidian space V_4, the radius vectors of which are determined by the vector field w. As seen, this space V_4 is a natural generalization of three-dimensional space of the ordinary hodograph. The space V_4 is usually called a generalized hodograph.

The concept of the generalized hodograph permits the construction of the variety of types of flows which were discussed at the very beginning.

It is generally accepted that a wave is called of the order q, or a q-wave, namely, in particular, a simple wave ($q = 1$), a double wave ($q = 2$) or a triple wave ($q = 3$), because the potential flow* of a compressible ideal isentropic gas may be mapped in the space of the generalized hodograph onto the qth surface, namely onto a line, onto a surface proper or onto a hypersurface, respectively.

It is clear that in all these cases we deal with a degenerate representation in the hodograph: some four-dimensional region of the motion space R_4 reflects onto the region V_q of the space of the generalized hodograph V_4 with the measure $q < 4$. Analytically this circumstance manifests itself by the fact that the functional determinant, or the Jacobian of the mapping functions $w_m(x^1, x^2, x^3, x^4)$, is identically equal to zero $|\partial w_m / \partial x^l| = 0$ (evidently only in the part of motion space which is

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* In an earlier paper (4) it was shown that in the case of simple nonsteady waves the potential nature of the flow follows from its very definition. With respect to other q-waves, this property of flow potential is introduced, as a rule, as an additional condition for the simplification of the problem.

occupied by the considered q-wave). Also the matrix of the Jacobian is of rank q.

Consequently, a q-wave may also be determined in the following way: it is a potential flow, whose vector field w has only q independent components w_m ($q < m = 4$) in an orthogonal Cartesian system, out of m possible components.

Note. It is recalled that the two first exact solutions of nonlinear equations of gasdynamics, that obtained in 1860 by Riemann [1], and the other in 1907 by Prandtl and Meyer [2], belong to the above-mentioned spatial forms of flows. However, these solutions do not nearly exhaust the scope of the classes of motion considered here. The flows of the Prandtl and Meyer type form only a small subclass of simple waves; even in a very general interpretation, when both "Riemann invariants" are variable, these also represent a somewhat limited subclass of double waves, because they depend only on two variables: namely, on a spatial, for example $x^1 \equiv x$, and on a time variable, $x^4 \equiv v^0 t$.

In the course of recent years the study of various wave classes has undoubtedly progressed. The problem of simple waves in particular, on the basis of several recent papers [3-7], may be considered, in some respects, to be almost solved. Significant and interesting results are obtained also in the theory of double waves, steady [3,8] and nonsteady [9-11]. The triple waves are considered in [12].

5. As noted above, the study of the general properties of double waves reduces basically to the analysis of the geometrical construction of the mapping of the physical space onto the surface of the corresponding degenerate hodograph. In the orthogonal coordinate systems (x^k) and $(w^m = w_m)$ this mapping is determined by some system of functions

$$w_m \equiv w_m(x^1, x^2, x^3, x^4) \qquad (m = 1, 2, 3, 4) \tag{5.1}$$

with the condition that the rank of the matrix $\|\partial w_m / \partial x^k\|$ is equal to two. From this condition it follows that the system (5.1) does not permit any single-valued transformation; to be more exact, to every point P of the hodograph of the double wave there corresponds in the physical space $R_4\{x^k\}$ some surface Π. On every such surface Π, at all its points x^k, we have, by definition, w = const and, hence, w_m = const, c = const, ρ = const, p = const. For the sake of simplicity, we shall call these surfaces "isodynamical surfaces".

In what follows we would also need, beside Equations (5.1), the parametric hodograph equations of the double wave under consideration

$$w_m \equiv w_m(\xi^1, \xi^2) \qquad (m = 1, 2, 3, 4) \tag{5.2}$$

where the rank of the matrix $||\partial w_m / \partial \xi^{\alpha}||$ is equal to two.

Note that the parameters ξ^1 and ξ^2 serve at the hodograph surface of the double wave considered, as the curvilinear coordinates ξ^{α} (α = 1, 2) of the points P. Consequently, ξ^1 and ξ^2 may also be considered to be the parameters which determine the two-parameter family of the isodynamical surface.

To simplify the investigation to follow it is convenient to introduce the Legendre function

$$\Phi \equiv x^m w_m - \varphi(x^1, x^2, x^3, x^4) \tag{5.3}$$

whose total differential $d\Phi$ on account of (2.3) does not contain differentials of the variables x^m. Then considering (5.2), we obtain

$$d\Phi = x^m dw_m = x^m \frac{\partial w_m}{\partial \xi^{\alpha}} d\xi^{\alpha} \tag{5.4}$$

From this equation it follows, first of all, that the Legendre function for the double wave depends only on the variables ξ^{α}:

$$\Phi \equiv \Phi(\xi^1, \xi^2) \tag{5.5}$$

and evidently, it also assumes a constant value on every isodynamical surface Π: secondly

$$\frac{\partial \Phi}{\partial \xi^{\alpha}} = x^m \frac{\partial w_m}{\partial \xi^{\alpha}} \qquad (\alpha = 1,2) \tag{5.6}$$

It is significant here that, according to (5.5) and (5.2), the free terms and coefficients of this system of equations, linear with respect to variables x^m, are functions of the parameters ξ^1 and ξ^2 only. Hence we have the following theorem:

Theorem 1. Isodynamical surfaces form in the motion space $R_4\{x^m\}$ the two-parameter family of surfaces Π, determined by the system of two linear equations of first order (5.6).

Further analysis of this system of equations shows that the geometrical form of the hodograph surface of double waves defines also the orientation of the surfaces Π. Indeed, the numerical values of the coefficients of these equations (5.6) define two vectors $W^m_{(\alpha)}(\xi^1, \xi^2)$ in the system of coordinates x^m of the motion space R_4:

$$W^m_{(\alpha)} \equiv \frac{\partial w^m}{\partial \xi^{\alpha}} = \frac{\partial w_m}{\partial \xi^{\alpha}} = \frac{\partial w_m}{\partial x^k} \frac{\partial x^k}{\partial \xi^{\alpha}} \qquad \begin{pmatrix} m = 1, 2, 3, 4 \\ \alpha = 1,2 \end{pmatrix} \tag{5.7}$$

which, because of Equations (5.6), are orthogonal to the investigated plane Π. But, on the other hand, taking into the consideration the form of these expressions as partial derivatives $\partial w^m / \partial \xi^\alpha$, it is clear that they form, in the hodograph space with the system of orthogonal Cartesian coordinates $w^m = w_m$, the components of two vectors, tangent at the point P of the hodograph surface corresponding to the coordinate line ξ^1 or ξ^2. Assuming further, for the purpose of simplification of the formulation of the final results, that in both systems of coordinates x^m and w^m the axes having the same numerical indices are parallel, we obtain the second theorem.

Theorem 2. Every isodynamic plane Π is orthogonal to the hodograph surface of a double wave at the corresponding point P of the hodograph.

6. Based on the theorems obtained, we may, using the system of equations (5.6), find an interesting analytic expression for the multiple-valued dependence of the variables x^m on the curvilinear coordinates ξ^α. It is easily shown that one of the possible forms of this expression is the following system which contains two arbitrary parameters η and ζ:

$$x^m \equiv f^m(\xi^1, \xi^2; \eta, \zeta) = g^{\beta\gamma} \frac{\partial w^m}{\partial \xi^\beta} \frac{\partial \Phi}{\partial \xi^\gamma} + \eta \pi_{(1)}{}^m + \zeta \pi_{(2)}{}^m \qquad (6.1)$$

where the contra- and covariant coordinates of the metric tensor of the hodograph surface satisfy the known relationships:

$$g^{\gamma\beta} g_{\alpha\beta} = \delta^\gamma_\alpha \begin{cases} 1 \ (\alpha = \gamma), \\ 0 \ (\alpha \neq \gamma), \end{cases} \qquad g_{\alpha\beta} = \frac{\partial w_m}{\partial \xi^\alpha} \frac{\partial w^m}{\partial \xi^\beta} \qquad (\alpha, \beta = 1,2) \qquad (6.2)$$

while the unit non-parallel vectors $\pi_{(1)}{}^m (\xi^1, \xi^2)$ and $\pi_{(2)}{}^m (\xi^1, \xi^2)$ are situated on the corresponding isodynamic plane Π. In order to prove the validity of Formula (6.1), it is sufficient to show that the first term on its right-hand side identically satisfies the system of equations (6.2). Indeed

$$x^m \frac{\partial w_m}{\partial \xi^\alpha} = g^{\beta\gamma} \frac{\partial w^m}{\partial \xi^\beta} \frac{\partial \Phi}{\partial \xi^\gamma} \frac{\partial w_m}{\partial \xi^\alpha} = g^{\beta\gamma} g_{\beta\alpha} \frac{\partial \Phi}{\partial \xi^\gamma} = \delta^\gamma_\alpha \frac{\partial \Phi}{\partial \xi^\gamma} = \frac{\partial \Phi}{\partial \xi^\alpha}$$

7. In Section 5 were derived the general properties of double waves independent of the equations of gasdynamics. These geometrical properties are a direct consequence of the special kinematic conditions by which the class of irrotational (potential) double waves is defined.

In the remainder of the investigation we use the equations of motion (2.9). Taking into consideration the parametric equations (5.2), we write the equation of motion in the form

$$T^{lm}\frac{\partial w_m}{\partial \xi^\alpha}\frac{\partial \xi^\alpha}{\partial x^l}=0 \qquad (7.1)$$

which reveals the essential role of curvilinear coordinates ξ^a $(a = 1,2)$.

We are interested here in the general "intrinsic" properties of double waves, i.e. the properties which do not depend on the special conditions of motion, that is, on the initial and boundary conditions. Keeping this in mind we shall try to express all the terms of Expression (7.1) in terms of functions of the curvilinear hodograph coordinates ξ^a of the double wave. First we shall do this with regard to the derivatives $\partial\xi^a/\partial x^l$. Here we make use of Equations (5.6) again, differentiating both its parts with respect to x^k:

$$\frac{\partial^2\Phi}{\partial\xi^\alpha\partial\xi^\beta}\frac{\partial\xi^\beta}{\partial x^k}=\delta_k^m\frac{\partial w_m}{\partial\xi^\alpha}+x^m\frac{\partial^2 w_m}{\partial\xi^\alpha\partial\xi^\beta}\frac{\partial\xi^\beta}{\partial x^k}$$

where $x^m \equiv f^m(\xi^1, \xi^2; \eta, \zeta)$ according to (6.1). These equations may be represented in a more concise form

$$h_{\alpha\beta}\frac{\partial\xi^\beta}{\partial x^k}=\frac{\partial w_k}{\partial\xi^\alpha} \qquad \begin{pmatrix}\alpha=1,2\\ k=1,2,3,4\end{pmatrix} \qquad (7.2)$$

if we designate by one symbol $h_{\alpha\beta}$ the covariant coordinates of the symmetric tensor

$$h_{\alpha\beta}\equiv\frac{\partial^2\Phi}{\partial\xi^\alpha\partial\xi^\beta}-f^m\frac{\partial^2 w_m}{\partial\xi^\alpha\partial\xi^\beta}, \qquad h_{\alpha\beta}=h_{\beta\alpha} \qquad (\alpha,\beta=1,2) \qquad (7.3)$$

For every arbitrarily chosen value of index k, Formulas (7.2) represent a system of two linear equations with two unknowns $\partial\xi^\beta/\partial x^k$. We shall denote by $H^{\alpha\beta}$ the algebraic addition $h_{\alpha\beta}$. From the foregoing we have the known relations

$$H^{\alpha\beta}=g\varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}h_{\gamma\delta}, \qquad \varepsilon^{11}=\varepsilon^{22}=0$$
$$\varepsilon^{12}=-\varepsilon^{21}=1/\sqrt{g}, \quad g\equiv|g_{\alpha\beta}| \qquad (7.4)$$

Using that expression, with the condition that $h \equiv |h_{\alpha\beta}| \neq 0$, we may represent the solution of the system (7.2) in the form

$$\frac{\partial\xi^\beta}{\partial x^k}=\frac{H^{\alpha\beta}}{h}\frac{\partial w_k}{\partial\xi^\alpha} \qquad \begin{pmatrix}\beta=1,2\\ k=1,2,3,4\end{pmatrix} \qquad (7.5)$$

Expressions (6.1) and (7.3) to (7.5) serve to eliminate effectively the variables x^k from Equation (7.1). In this manner we arrive at the equation basic for the further investigation:

$$T^{lm}\frac{\partial w_m}{\partial\xi^\alpha}\frac{\partial w_l}{\partial\xi^\beta}H^{\alpha\beta}=0 \tag{7.6}$$

where

$$H^{\alpha\beta}=g\varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}\left[\frac{\partial^2\Phi}{\partial\xi^\gamma\partial\xi^\delta}-\left(g^{\sigma\omega}\frac{\partial w^m}{\partial\xi^\sigma}\frac{\partial\Phi}{\partial\xi^\omega}+\eta\pi_{(1)}{}^m+\zeta\pi_{(2)}{}^m\right)\frac{\partial^2 w_m}{\partial\xi^\gamma\partial\xi^\delta}\right] \tag{7.7}$$

But, taking into account (6.2), for the Christoffel symbol of the first order we obtain

$$[\alpha\beta,\gamma]=\tfrac{1}{2}\left(\frac{\partial g_{\beta\gamma}}{\partial\xi^\alpha}+\frac{\partial g_{\alpha\gamma}}{\partial\xi^\beta}-\frac{\partial g_{\alpha\beta}}{\partial\xi^\gamma}\right)=\frac{\partial^2 w_m}{\partial\xi^\alpha\partial\xi^\beta}\frac{\partial w^m}{\partial\xi^\gamma}$$

Using this expression, we readily find in Equation (7.7) in square brackets, among others, the expression which is reducible to the covariant derivative

$$\frac{\partial^2\Phi}{\partial\xi^\gamma\partial\xi^\delta}-g^{\sigma\omega}[\gamma\delta,\sigma]\frac{\partial\Phi}{\partial\xi^\omega}=\frac{\partial^2\Phi}{\partial\xi^\gamma\partial\xi^\delta}-\Gamma^{\omega}_{\gamma\delta}\frac{\partial\Phi}{\partial\xi^\omega}=\nabla_\delta\left(\frac{\partial\Phi}{\partial\xi^\gamma}\right)=\nabla_{\gamma\delta}\Phi=\nabla_{\delta\gamma}\Phi$$

Taking into consideration the relationships obtained above, we can now state the basic equation (7.6) in its final form

$$T^{lm}\frac{\partial w_m}{\partial\xi^\alpha}\frac{\partial w_l}{\partial\xi^\beta}\varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}\left[\nabla_{\gamma\delta}\Phi-(\eta\pi_{(1)}{}^m+\zeta\pi_{(2)}{}^m)\frac{\partial^2 w_m}{\partial\xi^\gamma\partial\xi^\delta}\right]=0 \tag{7.8}$$

It is essential here that the values of the parameters η and ζ, which appear in this equation, are absolutely arbitrary. This shows that Equation (7.8) is equivalent to the system of three independent differential equations, the first of which may be written

$$T^{lm}\frac{\partial w_m}{\partial\xi^\alpha}\frac{\partial w_l}{\partial\xi^\beta}\varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}\nabla_{\gamma\delta}\Phi=0 \tag{7.9}$$

while two others are obtained by replacing the covariant derivative $\nabla_{\gamma\delta}\Phi$ by $\pi_{(1)}{}^m\partial^2 w_m/\partial\xi^\gamma\partial\xi^\delta$ and $\pi_{(2)}{}^m\partial^2 w_m/\partial\xi^\gamma\partial\xi^\delta$ respectively.

In the very structure of the equations of this system and in their physical interpretation an essential difference is easily noticeable. The last two of the given equations deal exclusively with the geometrical structure of the hodograph of a double wave (we shall recall that $\pi_{(1)}{}^m$ and $\pi_{(2)}{}^m$ denote unit vectors, orthogonal to the surface of this hodograph). For a detailed study of the flows themselves and of their particular features defined by the Legendre function Φ, it is necessary to turn to Equation (7.9). However, the role of the differential equation (7.9) in the theory of double waves is not confined to this alone:

using it, we may, among others, easily show some interesting, quite general and basic properties of double waves.

With this in mind and noting that Equation (7.9) is linear, with partial derivatives of the second order, we shall write immediately the corresponding equation of its characteristics:

$$T^{lm}\frac{\partial w_m}{\partial \xi^\alpha}\frac{\partial w_l}{\partial \xi^\beta}d\xi^\alpha d\xi^\beta = 0$$

Hence it follows that along the characteristics themselves we have

$$T^{lm}\,dw_m\,dw_l = 0 \tag{7.10}$$

Writing this equation in an expanded form, we find, according to (2.8)

$$\pm c\sqrt{\tau^{lm}\,dw_m\,dw_l} = u^k\,dw_k$$

Taking into account (2.7), (2.4) and (1.5), we obtain

$$ds \equiv \sqrt{(dv_1)^2 + (dv_2)^2 + (dv_3)^2} = \pm\frac{c}{\rho}\,d\rho = \pm\frac{2dc}{\varkappa - 1} \tag{7.11}$$

where the symbol ds means the differential of an arc of projection of the given characteristics in the usual three-dimensional vector space with Cartesian coordinates v_1, v_2, v_3. In this manner, the result obtained allows us to formulate a basic theorem, which reveals a close connection between the structure of hodographs of double and simple waves.

Theorem 3. On the surface of a hodograph of an arbitrary double wave the same "condition of combined motion" (7.11) holds along its characteristics, which imposes well-known restrictions (3.5) upon the freedom of choice of the hodograph line of a simple wave.

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